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Eigenvalue Sensitivity Analysis of Planar Frames with Variable Joint and Support Locations

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Various sensitivity equations are developed in this study for eigenvalue sensitivity analysis of planar frames with variable joint and support locations. The emphasis is placed on the derivation of the sensitivity equations by the continuum approach. A variational form of an eigenvalue equation is first derived in which all of the quantities are expressed with respect to the local coordinate system. The material derivative of this variational form is then sought to account for the changes in the member's length and orientation resulting from the perturbation of joint and support locations. Finally, eigenvalue sensitivity equations are formulated in either domain quantities (by the domain method) or boundary quantities (by the boundary method). Examples are presented to validate these derived sensitivity equations with analytical solutions as well as numerical results. It is concluded that the sensitivity equation derived by the boundary method is computationally more efficient but less accurate than that of the domain method. Nevertheless, both of them are superior to the conventional discrete analytical method in terms of computational efficiency.

Nomenclature

A_i, E_i, I_i	= cross-sectional area, Young's modulus, and moment of inertia of member i, respectively
b	= design variable
J, $\dot{J}$	= a line functional J and its total shape derivative
$[K]$, $[M]$	= global stiffness and mass matrices
$[K_i]$, $[M_i]$	= elementary stiffness and mass matrices
l_i, θ_i, ρ_i	= length, orientation angle, and mass density of member i, respectively
m_i, n_i	= cosine and sine of member i
N	= total number of members
$\bar{N}$	= number of members connected at a joint
NE	= total number of finite elements
P_i, S_i, M_i	= internal axial force, shear force, and bending moment of member i, respectively
(s_i, θ_i)	= length and orientation of member i
$[T_i]$	= elementary transformation matrix of element i
U_i, W_i	= eigenfunctions of member i in the global coordinate system
u_i, w_i	= eigenfunctions in the local coordinate system of member i
$\dot{u}_i$, $\dot{w}_i$	= total shape derivatives of eigenfunctions in the local coordinate system
$\mathring{u}_i$, $\mathring{w}_i$	= shape derivatives of eigenfunctions in which the orientation θ_i of the member i remains unchanged
$u_{,\tau}$, $w_{,\tau}$	= relative shape derivatives of eigenfunctions in the local coordinate system
v_i	= design velocity function for member i
(X, Y)	= location of a joint in the global coordinate system
$\{X_i\}$, $\{x_i\}$	= eigenvectors in the global and local coordinate systems
$\dot{\lambda}$	= total shape derivative of eigenvalue λ
τ	= monitoring parameter of the location variation
π	= weak variational functional of an eigenvalue problem
Φ_i, Ψ_i	= testing functions in the global coordinate system
ϕ_i, φ_i	= testing functions in the local coordinate system
Ω_i, R_i	= slopes at joint i

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Introduction

DESIGN sensitivity analysis aims to find the effect of changes of design variables on the responses of structural systems. In the literature, design sensitivity analysis has been used in a variety of enginering applications[1-3] and plays a vital role in the iterative optimum design schemes.[2,4,5] In particular, eigenvalue sensitivity analysis can be used to assess the design trends due to structural modifications,[6] reduce computational effort in structural optimization with frequency constraints,[7,8] and improve the structural model.[9,10]

Design sensitivity analysis has been the subject of intensive study in recent years. The notable advances to date are in the area of shape sensitivity analysis, which considers the shape or configuration parameters as design variables. In general, two groups of shape design variables can be recognized in engineering applications. One is composed of the contour profiles of a continuous domain, which can be described as continuous functions. The other is composed of the joint or support locations of a skeletal structure, such as beam, truss, frame, etc., which are simply distinct parameters.

The methods for shape sensitivity analysis can generally be divided into two categories: the continuum approach and the discrete approach. In the continuum approach, the sensitivity equation is derived based on the weak variational form of the governing differential equation. On the other hand, the discrete approach derives sensitivity equations based on the matrix equation obtained from spatial discretization of the governing differential equation. Several methods, in both approaches, have been reported and evaluated in the literature. However, owing to the nature of problem formulation, methods of the continuum approach have been perferably applied to the problems with continuous domains. Only a few attempts[11-15] have used the continuum approach to derive the sensitivity equations of skeletal structures.

Garstecki and Mroz[11] applied the method of generalized calculus of variations to derive sensitivity equations of static responses of structures with respect to the support stiffness, position, and orientation in continuous beams. Recently, Hou and Chuang[12,13] developed eigenvalue and eigenvector sensitivity equations of continuous beams subjected to the variations of support locations. Both the domain and the boundary methods were employed in their derivations. Their numerical results show that the sensitivity equations derived by the continuum approach have the same accuracy but are computationally more efficient than those derived by the discrete approach. A similar conclusion was also drawn by Choi and Twu[14] for static analysis. In another work, Choi and Twu[15] used the domain method to derive sensitivity equations of static responses of built-up structures in which the joint locations are considered as design variables.

The domain and the boundary methods mentioned earlier are the major tools in the continuum approach for shape sensitivity analysis.[16-19] The sensitivity equations derived by the domain method are usually expressed in the form of integrals. On the other hand, shape sensitivity equations derived by the boundary method are usually expressed as algebraic equations in terms of quantities defined at the boundaries. It is known that the computation of sensitivity coefficients using equations derived by the domain method is less economical but more accurate than that by the boundary method.

The purpose of this study is to extend the work done previously for one-dimensional continuous beams[12] to planar frame structures. Both joint and support locations will be allowed to vary in this study. Clearly, the configuration of a planar frame can be represented by the global coordinates of joints (including supports). Thus, it is logical to select the joint locations as shape design variables for sensitivity analysis. However, as will be shown later, the free vibration equations are usually expressed in terms of the lengths and orientations of members rather than the joint coordinates. For this reason, the local quantities (l_i, θ_i) of each member i will be selected as the intermediate variables in the following derivation. Nevertheless, the joint coordinates can be related to the local quantities in a one-to-one fashion. For example, assume the end points of the member i are (X_1, Y_1) and (X_2, Y_2), as shown in Fig. 1. The length and the orientation of this member may be expressed in terms of joint coordinates as

$$l_i = \sqrt{(X_2 - X_1)^2 + (Y_2 - Y_1)^2}$$
$$\theta_i = \cos^{-1}\left(\frac{X_2 - X_1}{l_i}\right) \quad \text{or} \quad \theta_i = \sin^{-1}\left(\frac{Y_2 - Y_1}{l_i}\right) \tag{1}$$

The challenge of this new work is to develop the basic relations of the shape variations including not only the length but also the orientation of each member.

As in Ref. 12, both the domain and the boundary methods will be applied herein to derive shape sensitivity equations using the concept of the material derivative. At the end of the derivation, three examples, namely, a cantilever beam, a 4-member frame, and a 19-member frame, are presented to investigate issues associated with numerical implementation. Since exact eigenfunctions of a cantilever beam are known, the first example provides an opportunity to investigate the accuracy of the derived sensitivity equations analytically. The other two examples have to rely on the finite element method to support the eigenvalue analysis.

Free Vibration of a Planar Frame

Let a planar frame consist of N straight members, each of which is confined by a pair of joints. A local coordinate system (s,θ) can be introduced for each member, in which the s axis is the member axis and the θ axis is the orientation angle of the member. Accordingly, the free vibration of each member of a planar frame can be decomposed into axial and lateral components. More specifically, the governing differential equations of free vibration of a planar frame structure with N members can be stated as follows:

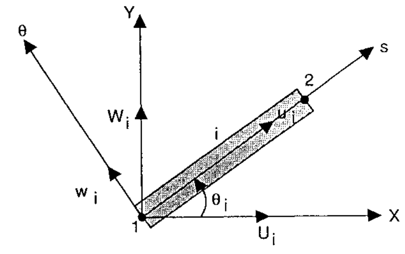

Fig. 1 Coordinate systems of a member in two-dimensional space.

$$E_iA_iu_i'' + \lambda\rho_iA_iu_i = 0, \qquad i = 1, 2, \ldots, N \tag{2}$$

and

$$E_iI_iw_i'''' - \lambda\rho_iA_iw_i = 0, \qquad i = 1, 2, \ldots, N \tag{3}$$

The $'$ denotes differentiation along the s axis. Equation (2) describes an axial vibration where $u_i(s)$ is the corresponding eigenfunction. Similarly, Eq. (3) defines a lateral vibration with $w_i(s)$ as the corresponding eigenfunction. To uniquely specify u_i and w_i, the eigenfunctions should satisfy the normalization condition

$$\sum_{i=1}^{N}\int_0^{l_i} \rho_iA_i(u_i^2 + w_i^2)\, ds = 1 \tag{4}$$

The weak variational form π of Eqs. (2) and (3) can be stated as

$$\begin{aligned}\pi &= 0\\ &= \sum_{i=1}^{N}\int_0^{l_i} [(E_iA_iu_i'' + \lambda\rho_iA_iu_i)\phi_i + (\lambda\rho_iA_iw_i - E_iI_iw_i'''')\varphi_i]\, ds\\ &= \sum_{i=1}^{N}\int_0^{l_i} [\lambda\rho_iA_i(u_i\phi_i + w_i\varphi_i) - E_i(A_iu_i'\phi_i' + I_iw_i''\varphi_i'')]\, ds\\ &+ \sum_{i=1}^{N}(E_iA_iu_i'\phi_i - E_iI_iw_i'''\varphi_i + E_iI_iw_i''\varphi_i')\Big|_0^{l_i}\end{aligned} \tag{5}$$

Note that all quantities in the preceding equations are defined in terms of the local coordinate system of the corresponding member. Next, to define the boundary conditions properly, the boundary terms in Eq. (5) should be expressed in the global coordinate system. Equations that relate the local eigenfunctions (u_i, w_i, u_i', w_i') to their counterparts in the global coordinate system (U_i, W_i, U_i', W_i') are defined as

$$\begin{bmatrix} u_i \\ w_i \end{bmatrix} = \begin{bmatrix} m_i & n_i \\ -n_i & m_i \end{bmatrix}\begin{bmatrix} U_i \\ W_i \end{bmatrix} \tag{6}$$

and

$$\begin{bmatrix} u_i' \\ w_i' \end{bmatrix} = \begin{bmatrix} m_i & n_i \\ -n_i & m_i \end{bmatrix}\begin{bmatrix} U_i' \\ W_i' \end{bmatrix} \tag{7}$$

where $m_i = \cos\theta_i$ and $n_i = \sin\theta_i$, in which θ_i is defined as the orientation angle between the global X axis and the local s axis of member i as shown in Fig. 1. Note that Eqs. (6) and (7) are also applicable for testing functions $(\phi_i, \varphi_i, \phi_i', \varphi_i')$ in the local coordinate system as well as their counterparts in the global coordinate system $(\Phi_i, \Psi_i, \Phi_i', \Psi_i')$.

With the aid of the previous relations, the boundary terms in Eq. (5) can be rewritten as

$$\sum_{i=1}^{N} [(P_i m_i - S_i n_i)\Phi_i + (P_i n_i + S_i m_i)\Psi_i + M_i \Omega_i] \Big|_0^{l_i} \tag{8}$$

where

$$P_i = E_i A_i u_i' \qquad S_i = -E_i I_i w_i''' \qquad M_i = E_i I_i w_i'' \tag{9}$$

and Ω_i is the slope at the same point,

$$\Omega_i = -n_i \Phi_i' + m_i \Psi_i' \tag{10}$$

It should be noted that Φ_i, Ψ_i, and Ω_i have unique values at each joint. Therefore, either the kinematic boundary conditions can be defined at the support joints or the natural boundary conditions that require the balance of internal forces at the interior joints can be specified. The natural boundary conditions at each interior joint are, in fact, given as

$$\sum_{i=1}^{\bar{N}} (P_i m_i - S_i n_i) = 0 \qquad \sum_{i=1}^{\bar{N}} (P_i n_i + S_i m_i) = 0 \qquad \sum_{i=1}^{\bar{N}} M_i = 0 \tag{11}$$

Concept of the Material Derivative

The concept of the material derivative has been proven to be very useful in shape sensitivity analysis.[4,19] In this work, this concept will be extended to derive eigenvalue sensitivity equations of planar frames. Before doing so, however, a brief introduction of this concept is in order.

The shape variation of any point in a varied domain may be viewed as a continuous transformation phenomenon between its final location x^* and the original one x. The position vector of the point at any intermediate stage can be expressed as

$$x(\tau) = x(0) + \tau v(x) \tag{12}$$

where x is the position vector and τ is a monitoring parameter ranging from 0 to 1 with $x(0) = x$ and $x(1) = x^*$. Note that only the linear variation is retained in Eq. (12). Mathematically, the term $v(x)$ is defined as $\dot{x} = v(x) = [dx(\tau)/d\tau]$ evaluated at $\tau = 0$ and is called the design velocity function.[4] As an example, the variations of a joint location (X, Y) can be given as

$$X(\tau) = X + \tau\dot{X} \qquad Y(\tau) = Y + \tau\dot{Y} \tag{13}$$

where the total shape derivatives of $\dot{X}$ and $\dot{Y}$ are identical to the perturbations of joint location, i.e.,

$$\dot{X} = X^* - X = \Delta X \qquad \dot{Y} = Y^* - Y = \Delta Y \tag{14}$$

With the preceding definitions, one can further obtain the shape derivative of the orientation θ_i as

$$\dot{\theta}_i = -\frac{n_i}{l_i}(\dot{X}_2 - \dot{X}_1) + \frac{m_i}{l_i}(\dot{Y}_2 - \dot{Y}_1) \tag{15}$$

where $\dot{X}_1$, $\dot{Y}_1$, $\dot{X}_2$, and $\dot{Y}_2$ are the position variations of the global coordinates of end joints 1 and 2 of member i, respectively.

The definition of Eq. (12) can be directly extended to any domain-dependent function. In any stage of domain transformation, a function $z(x)$ can be represented as $z[x(\tau),\tau]$ whose shape derivative at $\tau = 0$ is given as

$$\begin{aligned} \dot{z}(x) &= \dot{z}[x(\tau),\tau]\big|_{\tau=0} \\ &= \frac{dz}{d\tau}\bigg|_{\tau=0} \\ &= \lim_{\tau\to 0} \frac{z(x+\tau v,\tau) - z(x)}{\tau} \\ &= \frac{\partial z}{\partial \tau} + \frac{\partial z}{\partial x}\cdot v \\ &= z_{,\tau} + \nabla z \cdot v \end{aligned} \tag{16}$$

where the subscript $,\tau$ and ∇ denote the partial time and spatial derivatives, respectively. The last identity states that the total shape variation of a domain-dependent function is a combination of the variation of the function itself and the variation induced by the perturbations of locations of points in the domain. The detailed discussion of Eq. (16) can be found in Ref. 4, in which $z_{,\tau}$ is called the relative shape derivative and $\dot{z}$ is called the total shape derivation of function $z(x)$.

The last equation has been applied to find the eigenvalue and eigenvector sensitivities of a continuous beam with variable support locations,[12,13] in which only the length between a pair of supports is considered as a design variable. In this study, however, the concern is a planar frame whose configuration is defined not only by the length between a pair of joints but also by its orientation. Therefore, Eq. (16) should be applied herein with modifications.

Taking the total shape derivatives of eigenfunctions u_i and w_i in Eq. (6) yields

$$\begin{aligned} \begin{bmatrix} \dot{u}_i \\ \dot{w}_i \end{bmatrix} &= \begin{bmatrix} \dot{m}_i & \dot{n}_i \\ -\dot{n}_i & \dot{m}_i \end{bmatrix} \begin{bmatrix} U_i \\ W_i \end{bmatrix} + \begin{bmatrix} m_i & n_i \\ -n_i & m_i \end{bmatrix} \begin{bmatrix} \dot{U}_i \\ \dot{W}_i \end{bmatrix} \\ &= \dot{\theta}_i \begin{bmatrix} -n_i & m_i \\ -m_i & -n_i \end{bmatrix} \begin{bmatrix} U_i \\ W_i \end{bmatrix} + \begin{bmatrix} m_i & n_i \\ -n_i & m_i \end{bmatrix} \begin{bmatrix} \dot{U}_i \\ \dot{W}_i \end{bmatrix} \\ &= \dot{\theta}_i \begin{bmatrix} w_i \\ -u_i \end{bmatrix} + \begin{bmatrix} \dot{\mathfrak{u}}_i \\ \dot{\mathfrak{w}}_i \end{bmatrix} \end{aligned} \tag{17}$$

where $\dot{\mathfrak{u}}_i$ and $\dot{\mathfrak{w}}_i$ are defined as

$$\begin{bmatrix} \dot{\mathfrak{u}}_i \\ \dot{\mathfrak{w}}_i \end{bmatrix} = \begin{bmatrix} m_i & n_i \\ -n_i & m_i \end{bmatrix} \begin{bmatrix} \dot{U}_i \\ \dot{W}_i \end{bmatrix} \tag{18}$$

It should be noted that $\dot{u}_i$ and $\dot{w}_i$ are completely different from $\dot{\mathfrak{u}}_i$ and $\dot{\mathfrak{w}}_i$. The former are the total shape derivatives of eigenfunctions whereas the latter are the shape derivatives of eigenfunctions in which the orientation of the member θ_i remains unchanged. Equations (17) and (18) clearly indicate that the domain variation of a function pertaining to a planar frame is composed of two parts. One is related to the effect of orientational variation $\dot{\theta}_i$, and the other is the domain variation $\dot{\mathfrak{u}}_i$ and $\dot{\mathfrak{w}}_i$ along the local coordinate system. Furthermore, with the aid of Eq. (16), the total shape derivatives of u_i and w_i can also be represented in terms of the relative shape derivatives $u_{i,\tau}$ and $w_{i,\tau}$ as

$$\begin{bmatrix} \dot{u}_i \\ \dot{w}_i \end{bmatrix} = \dot{\theta}_i \begin{bmatrix} w_i \\ -u_i \end{bmatrix} + \begin{bmatrix} u_{i,\tau} \\ w_{i,\tau} \end{bmatrix} + v_i \begin{bmatrix} u_i' \\ w_i' \end{bmatrix} \tag{19}$$

where v_i is the design velocity function defined along the s axis of the local coordinate system. In the same manner, the total shape derivatives of other quantities such as u_i', w_i', and w_i'' can be derived. These are stated in the Appendix for reference.

Next, one may proceed to find the total shape derivative of a typical line functional defined over a straight line in a two-dimensional space as

$$J(u, w, u', w'', \tau) = \int_0^l f(u, w, u', w'', \tau)\, \mathrm{d}s \tag{20}$$

where functions $u(s)$ and $w(s)$ are defined with respect to the local coordinate system. Its total shape derivative can be shown in terms of $\dot{\theta}$, $\dot{u}$, and $\dot{w}$ as

$$\dot{J} = \int_0^l \left[\frac{\partial f}{\partial \tau} + \left(\frac{\partial f}{\partial u} w - \frac{\partial f}{\partial w} u + \frac{\partial f}{\partial u'} w' - \frac{\partial f}{\partial w''} u'' \right) \dot{\theta} + \frac{\partial f}{\partial u} \dot{u} + \frac{\partial f}{\partial w} \dot{w} + \frac{\partial f}{\partial u'} \dot{u}' + \frac{\partial f}{\partial w''} \dot{w}'' - \left(\frac{\partial f}{\partial u'} u' + 2 \frac{\partial f}{\partial w''} w'' - f \right) v' - \frac{\partial f}{\partial w''} w' v'' \right] \mathrm{d}s \tag{21}$$

or in terms of $\dot{\theta}$, $u_{,\tau}$, and $w_{,\tau}$ as

$$\dot{J} = \int_0^l \left[\frac{\partial f}{\partial \tau} + \left(\frac{\partial f}{\partial u} w - \frac{\partial f}{\partial w} u + \frac{\partial f}{\partial u'} w' - \frac{\partial f}{\partial w''} u'' \right) \dot{\theta} + \frac{\partial f}{\partial u} u_{,\tau} + \frac{\partial f}{\partial w} w_{,\tau} + \frac{\partial f}{\partial u'} u'_{,\tau} + \frac{\partial f}{\partial w''} w''_{,\tau} \right] \mathrm{d}s + (fv) \Big|_0^l \tag{22}$$

The detailed derivation of Eqs. (21) and (22) is given in the Appendix. These equations lay the ground work for the derivation of eigenvalue sensitivity equations of a frame structure with respect to joint or support locations. Equation (21) is the basic equation used in the domain method, whereas Eq. (22) is used in the boundary method.

Continuum Approach

In the following section, eigenvalue shape sensitivity equations are derived by the domain method as well as the boundary method. Although both methods follow a similar derivation procedure, they result in sensitivity equations of different forms.

Domain Method (DM)

Let the admissible functions ϕ_i and ψ_i in Eq. (5) be the eigenfunctions themselves, i.e., u_i and w_i. Consequently, the weak variational form of free vibration is then reduced to

$$\pi = 0 = \sum_{i=1}^N \int_0^{l_i} [\lambda \rho_i A_i (u_i^2 + w_i^2) - E_i (A_i u_i'^2 + I_i w_i''^2)]\, \mathrm{d}s \tag{23}$$

where the boundary terms have been dropped because the kinematic and natural boundary conditions are satisfied at the supports as well as at the interior joints.

Since Eq. (23) is a special case of Eq. (20), Eq. (21) can be applied here to find the total shape derivative of π. The resultant equation $\dot{\pi} = 0$ yields the following identity:

$$\dot{\lambda} \sum_{i=1}^N \int_0^{l_i} \rho_i A_i (u_i^2 + w_i^2)\, \mathrm{d}s = - \sum_{i=1}^N \int_0^{l_i} 2(E_i I_i u_i'' w_i'' - E_i A_i u_i' w_i') \dot{\theta}_i\, \mathrm{d}s - \sum_{i=1}^N \int_0^{l_i} \{ [\lambda \rho_i A_i (u_i^2 + w_i^2) + E_i A_i u_i'^2 + 3 E_i I_i w_i''^2] v_i' + 2 E_i I_i w_i'' w_i' v_i'' \}\, \mathrm{d}s - \sum_{i=1}^N \int_0^{l_i} 2(\lambda \rho_i A_i u_i \dot{u}_i - E_i A_i u_i' \dot{u}_i')\, \mathrm{d}s - \sum_{i=1}^N \int_0^{l_i} 2(\lambda \rho_i A_i w_i \dot{w}_i - E_i I_i w_i'' \dot{w}_i'')\, \mathrm{d}s \tag{24}$$

Incorporating this with the normalization condition of Eq. (4), the left side of Eq. (24) is simply equal to $\dot{\lambda}$. Furthermore, the last two terms in Eq. (24) can be simplified by means of integration by parts, yielding

$$- \sum_{i=1}^N \int_0^{l_i} 2(\lambda \rho_i A_i u_i \dot{u}_i - E_i A_i u_i' \dot{u}_i')\, \mathrm{d}s - \sum_{i=1}^N \int_0^{l_i} 2(\lambda \rho_i A_i w_i \dot{w}_i - E_i I_i w_i'' \dot{w}_i'')\, \mathrm{d}s = - \sum_{i=1}^N \int_0^{l_i} 2(\lambda \rho_i A_i u_i + E_i A_i u_i'') \dot{u}_i\, \mathrm{d}s - \sum_{i=1}^N \int_0^{l_i} 2(\lambda \rho_i A_i w_i - E_i I_i w_i'''') \dot{w}_i\, \mathrm{d}s + \sum_{i=1}^N 2(E_i A_i u_i' \dot{u}_i - E_i I_i w_i''' \dot{w}_i + E_i I_i w_i'' \dot{w}_i') \Big|_0^{l_i} = \sum_{i=1}^N 2(E_i A_i u_i' \dot{u}_i - E_i I_i w_i''' \dot{w}_i + E_i I_i w_i'' \dot{w}_i') \Big|_0^{l_i} \tag{25}$$

The integrals are dropped in Eq. (25) as the terms in the parentheses are exactly identical to the state equations presented by Eqs. (2) and (3). Further simplification of Eq. (25) is in order. The first step in this effort is to replace the terms $E_i A_i u_i'$, $-E_i I_i w_i'''$, and $E_i I_i w_i''$ by internal forces P_i, S_i, and M_i, respectively, as defined by Eq. (9). The next step is to convert the terms of $\dot{u}_i$, $\dot{w}_i$, and $\dot{w}_i'$ to their counterparts in the global coordinate system. In the end, one obtains the following equality:

$$\sum_{i=1}^N 2(E_i A_i u_i' \dot{u}_i - E_i I_i w_i''' \dot{w}_i + E_i I_i w_i'' \dot{w}_i') \Big|_0^{l_i} = \sum_{i=1}^N 2[(P_i m_i - S_i n_i) \dot{U}_i + (P_i n_i + S_i m_i) \dot{W}_i + M_i (\dot{R}_i + u_i' \dot{\theta}_i + v_i' w_i')] \Big|_0^{l_i} \tag{26}$$

where $\dot{R}_i$ is the shape derivative of the slope R_i, i.e., $R_i = -n_i U_i' + m_i W_i'$. Now, note that U_i, W_i, and R_i are equal to zero at each of the built-in supports. Hence, one has $\dot{U}_i = \dot{W}_i = \dot{R}_i = 0$ at the supports. Furthermore, since U_i, W_i, and R_i have unique values at the interior joints, the natural boundary conditions stated in Eq. (11) can be applied here to eliminate all of the terms in Eq. (26) except the last two,

$$(2 M_i u_i' \dot{\theta}_i + 2 M_i v_i' w_i') \Big|_0^{l_i}$$

Finally, after this manipulation, the sensitivity equation given by Eq. (24) can be rewritten in a shorter form as

$$\dot{\lambda} = - \sum_{i=1}^N \int_0^{l_i} 2(E_i I_i u_i'' w_i'' - E_i A_i u_i' w_i') \dot{\theta}_i\, \mathrm{d}s + \sum_{i=1}^N 2 E_i I_i w_i'' (u_i' \dot{\theta}_i + v_i' w_i') \Big|_0^{l_i} - \sum_{i=1}^N \int_0^{l_i} \{ [\lambda \rho_i A_i (u_i^2 + w_i^2) + E_i A_i u_i'^2 + 3 E_i I_i w_i''^2] v_i' + 2 E_i I_i w_i' w_i'' v_i'' \}\, \mathrm{d}s \tag{27}$$

which states that the shape derivative of λ is a linear functional of v_i and $\dot{\theta}_i$. Nevertheless, further simplification is still possible if the following integration by parts is performed:

$$\int_0^{l_i} E_i I_i u_i'' w_i'' \dot{\theta}_i\, \mathrm{d}s = E_i I_i w_i'' u_i' \dot{\theta}_i \Big|_0^{l_i} - \int_0^{l_i} E_i I_i w_i''' u_i' \dot{\theta}_i\, \mathrm{d}s \tag{28}$$

where $\dot{\theta}_i$ is independent of s. Thus, the first term of the first integral in Eq. (27) can be replaced by the previous equality to produce the following eigenvalue sensitivity equation:

$$\dot{\lambda} = \sum_{i=1}^{N}\int_0^{l_i} 2(E_iI_iu_i' w_i''' + E_iA_iu_i' w_i')\dot{\theta}_i \, ds + \sum_{i=1}^{N} 2E_iI_iw_i' w_i'' v_i' \Big|_0^{l_i} - \sum_{i=1}^{N}\int_0^{l_i} \{ [\lambda\rho_iA_i(u_i^2 + w_i^2) + E_iA_iu_i'^2 + 3E_iI_iw_i''^2]v_i' + 2E_iI_iw_i' w_i'' v_i'' \} \, ds \quad (29)$$

The term $\dot{\theta}_i$ in Eq. (29) has been shown to be related to the variation of joint locations by Eq. (15). However, the other term v_i, which is the velocity function defined along the s axis in each member, requires an investigation. Following the recommendation of Ref. 12, v_i is specified as a cubic polynomial:

$$v_i(s) = \left[1 - 3\left(\frac{s}{l_i}\right)^2 + 2\left(\frac{s}{l_i}\right)^3\right]v_1 + \left[3\left(\frac{s}{l_i}\right)^2 - 2\left(\frac{s}{l_i}\right)^3\right]v_2 \quad (30)$$

where $v_i(0) = v_1$ and $v_i(l_i) = v_2$ account for the axial movements of the end joints of member i. Mathematically, v_1 and v_2 can be represented by the following relations:

$$v_1 = m_i\dot{X}_1 + n_i\dot{Y}_1$$
$$v_2 = m_i\dot{X}_2 + n_i\dot{Y}_2 \quad (31)$$

The definition of $v_i(s)$ given by Eq. (30) provides a nice feature, $v_i'(0) = v_i'(l_i) = 0$, that eliminates the only boundary term from Eq. (29). As a result, the eigenvalue sensitivity equation of a planar frame derived by the domain method is given as

$$\dot{\lambda} = \sum_{i=1}^{N}\int_0^{l_i} 2(E_iI_iu_i' w_i''' + E_iA_iu_i' w_i')\dot{\theta}_i \, ds - \sum_{i=1}^{N}\int_0^{l_i} \{ [\lambda\rho_iA_i(u_i^2 + w_i^2) + E_iA_iu_i'^2 + 3E_iI_iw_i''^2]v_i' + 2E_iI_iw_i' w_i'' v_i'' \} \, ds \quad (32)$$

which is completely expressed in terms of line integrals.

Boundary Method (BM)

We now turn our attention to the boundary method. An eigenvalue sensitivity equation in terms of relative shape derivatives can be obtained by applying Eq. (22) to Eq. (23) as

$$\dot{\lambda} = -\sum_{i=1}^{N}\int_0^{l_i} 2(E_iI_iu_i'' w_i'' - E_iA_iu_i' w_i')\dot{\theta}_i \, ds - \sum_{i=1}^{N}\int_0^{l_i} 2[(\lambda\rho_iA_iu_iu_{i,\tau} - E_iA_iu_i' u_{i,\tau}') + (\lambda\rho_iA_iw_iw_{i,\tau} - E_iI_iw_i'' w_{i,\tau}'')] \, ds - \sum_{i=1}^{N} [\lambda\rho_iA_i(u_i^2 + w_i^2) - E_iA_iu_i'^2 - E_iI_iw_i''^2]v_i \Big|_0^{l_i} \quad (33)$$

where the normalization condition of Eq. (4) has been incorporated. Integration by parts of the second integral in Eq. (33) leads to the following result:

$$-\sum_{i=1}^{N}\int_0^{l_i} 2[(\lambda\rho_iA_iu_iu_{i,\tau} - E_iA_iu_i' u_{i,\tau}') + (\lambda\rho_iA_iw_iw_{i,\tau} - E_iI_iw_i'' w_{i,\tau}'')] \, ds = -\sum_{i=1}^{N}\int_0^{l_i} 2[(\lambda\rho_iA_iu_i + E_iA_iu_i'')u_{i,\tau} + (\lambda\rho_iA_iw_i - E_iI_iw_i'''')w_{i,\tau}] \, ds + \sum_{i=1}^{N} 2(E_iA_iu_i' u_{i,\tau} - E_iI_iw_i''' w_{i,\tau} + E_iI_iw_i'' w_{i,\tau}') \Big|_0^{l_i} \quad (34)$$

Two observations can be made here. First, the integral on the right-hand side is dropped because the terms in the parentheses are identical to the state equations of Eqs. (2) and (3). Second, the unknown boundary terms $u_{i,\tau}$, $w_{i,\tau}$, and $w_{i,\tau}'$ can be replaced by the known quantities based on the relations of Eqs. (17), (19), (26), and (A4):

$$\sum_{i=1}^{N} 2(E_iA_iu_i' u_{i,\tau} - E_iI_iw_i''' w_{i,\tau} + E_iI_iw_i'' w_{i,\tau}') \Big|_0^{l_i} = \sum_{i=1}^{N} 2(E_iA_iu_i' \dot{u}_i - E_iI_iw_i''' \dot{w}_i + E_iI_iw_i'' \dot{w}_i') \Big|_0^{l_i} - \sum_{i=1}^{N} [2(E_iA_iu_i'^2 - E_iI_iw_i''' w_i' + E_iI_iw_i''^2)v_i + 2E_iI_iw_i'' w_i' v_i'] \Big|_0^{l_i} = \sum_{i=1}^{N} 2E_iI_iw_i''(u_i'\dot{\theta}_i + v_i' w_i') \Big|_0^{l_i} - \sum_{i=1}^{N} [2(E_iA_iu_i'^2 - E_iI_iw_i''' w_i' + E_iI_iw_i''^2)v_i + 2E_iI_iw_i'' w_i' v_i'] \Big|_0^{l_i} \quad (35)$$

The combination of Eqs. (33–35) yields the basic equation for eigenvalue shape sensitivity:

$$\dot{\lambda} = \sum_{i=1}^{N} 2E_iI_iu_i' w_i'' \dot{\theta}_i \Big|_0^{l_i} - \sum_{i=1}^{N}\int_0^{l_i} 2(E_iI_iu_i'' w_i'' - E_iA_iu_i' w_i')\dot{\theta}_i \, ds - \sum_{i=1}^{N} [\lambda\rho_iA_i(u_i^2 + w_i^2) + E_iA_iu_i'^2 + E_iI_iw_i''^2 - 2E_iI_iw_i' w_i''']v_i \Big|_0^{l_i} \quad (36)$$

An alternative form of Eq. (36) can be obtained by integration by parts of the remaining integral in Eq. (36) as

$$\sum_{i=1}^{N}\int_0^{l_i} 2(E_iI_iu_i'' w_i'' - E_iA_iu_i' w_i')\dot{\theta}_i \, ds = \sum_{i=1}^{N}\int_0^{l_i} 2(E_iI_iw_i'''' u_i + E_iA_iu_i'' w_i)\dot{\theta}_i \, ds + \sum_{i=1}^{N} 2(E_iI_iw_i'' u_i' - E_iI_iu_iw_i''' - E_iA_iu_i' w_i)\dot{\theta}_i \Big|_0^{l_i} \quad (37)$$

where the integral is dropped because of the following identity

$$E_iI_iw_i'''' u_i + E_iA_iu_i'' w_i = (\lambda\rho_iA_iu_i + E_iA_iu_i'')w_i = 0 \quad (38)$$

which is proven by the state equations, Eqs. (2) and (3). Hence, the integral term in Eq. (36) can be completely replaced by the boundary terms given in Eq. (37). That is,

$$\dot{\lambda} = \sum_{i=1}^{N} 2(E_iA_iu_i' w_i + E_iI_iu_iw_i''')\dot{\theta}_i \Big|_0^{l_i} - \sum_{i=1}^{N} [\lambda\rho_iA_i(u_i^2 + w_i^2) + E_iA_iu_i'^2 + E_iI_iw_i''^2 - 2E_iI_iw_i' w_i''']v_i \Big|_0^{l_i} \quad (39)$$

It is understood that Eq. (39) is identical to Eq. (36) only when Eqs. (2) and (3) can be exactly satisfied. Hence, in conjunction with the finite element method, Eq. (39) can only provide approximate eigenvalue sensitivity coefficients. Numerical examples will be presented later to demonstrate the approximate nature of Eq. (39). Furthermore, it should be noted that the derived sensitivity equations, Eqs. (32) and (39), are valid not only for joint location variations but also for support location variations.

Discrete Approach

The discrete analytical method (DAM) is a discrete approach commonly used for sensitivity analysis in which the sensitivity equations are derived based on the discretized state equation. For the purpose of comparison, an eigenvalue sensitivity equation derived by the discrete analytical method will be presented hereafter.

The matrix equation of a planar frame under free vibration is given as

$$[K - \lambda M]\{X\} = \{0\} \tag{40}$$

which is the discretized version of Eqs. (2) and (3). $[K]$ and $[M]$ are assemblies of elementary stiffness and mass matrices $[K_i]$ and $[M_i]$ as

$$\begin{aligned} [K] &= \sum_{i=1}^{NE} [T_i]^T[K_i][T_i] \\ [M] &= \sum_{i=1}^{NE} [T_i]^T[M_i][T_i] \end{aligned} \tag{41}$$

Note that the matrices $[K_i]$, $[M_i]$, and $[T_i]$ are explicit functions of either the length or the orientation of element i. Let b denote the design variable pertaining to the X or Y coordinate of a joint. The eigenvalue sensitivity equation[20] then becomes

$$\begin{aligned} \frac{d\lambda}{db} &= 2\sum_{i=1}^{NE} \{x_i\}^T[T_i]\frac{d[T_i]^T}{db}([K_i] - \lambda[M_i])\{x_i\} \\ &+ \sum_{i=1}^{NE} \{x_i\}^T\left(\frac{d[K_i]}{db} - \lambda\frac{d[M_i]}{db}\right)\{x_i\} \end{aligned} \tag{42}$$

With the aid of Eq. (1), the gradients of the elementary stiffness, mass, and transformation matrices with respect to the design variables related to joint locations can be found without difficulty.

Examples

Three examples are collected here to investigate the numerical performance of the sensitivity equations derived in the preceding sections. The eigenvalues and eigenvectors are mainly evaluated by the finite element analysis. Each example has its own special emphasis. The first example is a cantilever beam whose eigenvalues and eigenvectors can be analytically derived. It provides an opportunity to show that both Eqs. (32) and (39) are capable of computing the exact eigenvalue sensitivity coefficients. The purpose of the second example, a four-member frame, is to demonstrate that both Eqs. (32) and (39) are valid for taking into account the location variations of not only joints but also supports. The last example, a 19-member frame, focuses on computational efficiency of the methods developed in this paper.

Cantilever Beam

A cantilever beam with its geometric parameters is depicted in Fig. 2. Only the lateral vibration is considered in this study. The eigensolutions[21] of this problem are given as

$$\lambda_r = \frac{(a_r L)^4}{L^4}\frac{EI}{\rho A}, \qquad r = 1, 2, 3, \ldots \tag{43}$$

and

$$X_r(s) = C_r\{\cosh(a_r s) - \cos(a_r s) - K_r[\sinh(a_r s) - \sin(a_r s)]\} \tag{44}$$

where the arc length s is varied between 0 and L, the amplitude

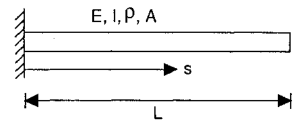

Fig. 2 Cantilever beam.

constant C_r is determined by the normalization condition, and the parameter K_r is determined by the following equation:

$$K_r = \frac{\cosh(a_r L) + \cos(a_r L)}{\sinh(a_r L) + \sin(a_r L)}, \qquad r = 1, 2, 3, \ldots \tag{45}$$

The terms $(a_r L)$ are the roots of a transcendental equation. The first three roots are given as

$$\begin{aligned} a_1 L &= 1.8751 \\ a_2 L &= 4.6941 \\ a_3 L &= 7.8548 \end{aligned} \tag{46}$$

In this example the orientation of the beam remains unchanged, that is, $\dot{\theta} = 0$. Therefore, the sensitivity equations, Eqs. (32) and (39), may be simplified in the following forms, respectively,

$$\dot{\lambda}_r = -\int_0^L [(\lambda_r \rho A X_r^2 + 3EI X_r''^2)v' + 2EI X_r' X_r'' v''] \, ds \qquad r = 1, 2, 3, \ldots \tag{47}$$

and

$$\dot{\lambda}_r = -(\lambda_r \rho A X_r^2 + EI X_r''^2 - 2EI X_r' X_r''')v \Big|_0^L \qquad r = 1, 2, 3, \ldots \tag{48}$$

where $v(s)$ is defined by Eq. (30) with $v_1 = 0$ and $v_2 = 1$. Substituting the exact values of the first three eigenvalues and eigenvectors, λ_r and X_r, into Eqs. (47) and (48), it can be shown that Eqs. (47) and (48) provide identical results. Furthermore, these sensitivity coefficients are the same as those obtained by directly taking the derivative of λ_r of Eq. (43), i.e.,

$$\frac{d\lambda_r}{dL} = -\frac{4(a_r L)^4}{L^5}\frac{EI}{\rho A}, \qquad r = 1, 2, 3, \ldots \tag{49}$$

This study thus confirms that the sensitivity equations expressed by Eqs. (47) and (48) are identical as long as the exact eigensolutions are used in the evaluation. In many engineering applications, however, the eigensolutions can only be obtained approximately. Therefore, it is necessary to study the effect of analysis inaccuracy on the accuracy of sensitivity analysis. To do so, the finite element solutions of various finite element meshes ranging from 1 to 30 elements are used to evaluate the sensitivity coefficients based on sensitivity equations derived by different methods, i.e., Eqs. (42), (47–49).

The results of this study are given in Figs. 3–7. The cantilever beam, 1.0 in. in length, is assumed to have a solid circular section with radius 0.1. Young's modulus and mass density are selected as 1,0000.0 psi and 1.0 slug/in.^{3}, respectively. Figure 3 shows the data associated with the convergence study of eigenvalue analysis. It is revealed that the one-element model is good enough to obtain an accurate first eigenvalue, whereas at least two- or three-element models are required to obtain satisfactory second or third eigenvalues. Figure 4 provides an opportunity to study the convergence of

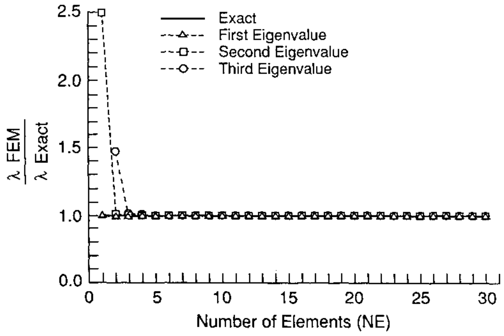

Fig. 3 Convergence study of eigenvalue analysis.

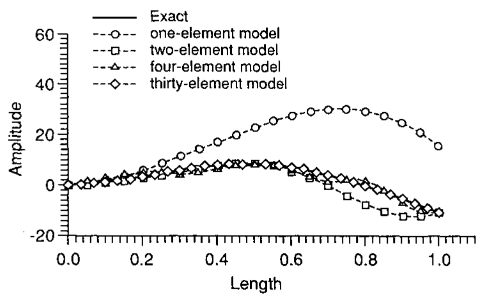

Fig. 4 Second mode shape of the cantilever beam.

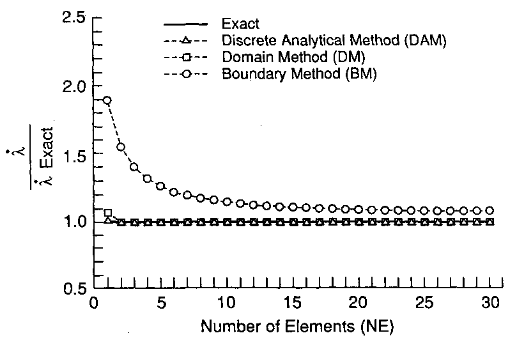

Fig. 5 Convergence study of the first eigenvalue sensitivity analysis.

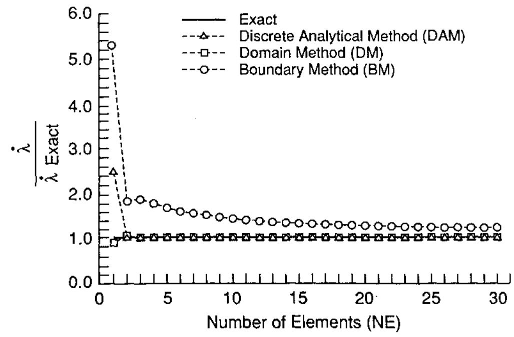

Fig. 6 Convergence study of the second eigenvalue sensitivity analysis.

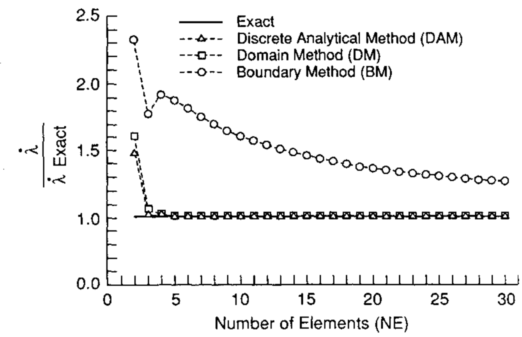

Fig. 7 Convergence study of the third eigenvalue sensitivity analysis.

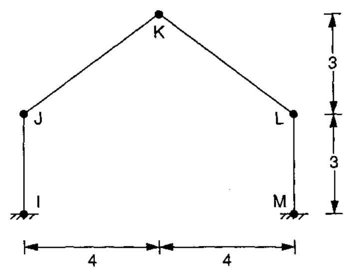

Fig. 8 Four-member frame.

eigenvectors. As shown in Fig. 4, the second eigenvector of the one-element model is quite different from the analytical one. As the number of elements is progressively increased, the eigenvector obtained by the finite element method converges to the exact one.

Figures 5–7 contain the data associated with the convergence of the first, second, and third eigenvalue sensitivities, respectively. The ordinate, y axis, in these figures is the ratio of the computed eigenvalue sensitivity coefficients to the analytical one, whereas the abscissa, x axis, is the number of elements used in the finite element eigenvalue analysis. The following conclusions can be drawn from these figures.

1) The sensitivity coefficients calculated by the domain method are more accurate than those found by the boundary method. In fact, the sensitivity coefficients calculated by the boundary method cannot converge to the exact values, even with a 30-element model.

The preceding conclusion is expected, though. As mentioned in Ref. 16, the boundary method does not, but the domain method does consider the variations of the across-element discontinuities in its derivation, which are inevitable in the finite element analysis.

2) Eigenvalue sensitivity coefficients of higher modes converge to the exact values slower than those of lower modes, particularly in the case of the boundary method. This may be attributed to the fact that, using the same mesh, the finite element method calculates lower modes more accurately than higher modes.

3) The convergence rate of the sensitivity coefficients calculated by the domain method is very similar to that of the eigenvalue analysis. In other words, the sensitivity coefficients calculated by the domain method will not be accurate if the eigenvalue analysis is not accurate. Therefore, the accuracy of the sensitivity coefficients calculated by the domain method may be used as an error indicator to measure the accuracy of the finite element analysis.

Four-Member Frame

The layout of a four-member frame and its geometric data are shown in Fig. 8. Each member has a solid circular section with a radius of 0.075 in. The Young's modulus and the mass density are given as 1,0000.0 psi and 0.25 slug/in.^{3}, respectively. Three finite element meshes with one, four, and eight elements in each member, respectively, are considered in this example. Two cases are studied here. In the first case, the location of support I is considered as the design variable. In the second case, the locations of joints J, K, and L are simultaneously considered as design variables. Tables 1 and 2 document the numerical results of the first three eigenvalue sensitivities. Note that the labels (X) and (Y) indicate the X and Y coordinates, respectively, of the joint being considered

Table 1 Eigenvalue sensitivity coefficients of four-member frame with respect to movements of support *I*

Mesh	Initial eigenvalues	CDM (*X*)	CDM (*Y*)	DAM (*X*)	DAM (*Y*)	DM (*X*)	DM (*Y*)	BM (*X*)	BM (*Y*)
1	$\lambda_1 = 1.587$	−0.197	0.628	−0.197	0.628	−0.197	0.632	−0.197	0.611
	$\lambda_2 = 6.168$	−1.049	1.764	−1.049	1.764	−1.049	1.757	−1.049	1.611
	$\lambda_3 = 33.322$	−2.998	5.786	−2.998	5.786	−2.992	5.746	−2.994	4.089
4	$\lambda_1 = 1.565$	−0.188	0.608	−0.188	0.608	−0.188	0.608	−0.188	0.608
	$\lambda_2 = 6.124$	−1.053	1.753	−1.053	1.753	−1.053	1.753	−1.053	1.753
	$\lambda_3 = 23.608$	−2.309	3.614	−2.309	3.614	−2.309	3.612	−2.309	3.608
8	$\lambda_1 = 1.565$	−0.188	0.608	−0.188	0.608	−0.188	0.608	−0.188	0.608
	$\lambda_2 = 6.123$	−1.053	1.753	−1.053	1.753	−1.053	1.753	−1.053	1.753
	$\lambda_3 = 23.590$	−2.307	3.611	−2.307	3.611	−2.307	3.611	−2.307	3.610

Table 2 Eigenvalue sensitivity coefficients of four-member frame with respect to simultaneous movements of joints *J*, *K*, and *L*

Mesh	Initial eigenvalues	CDM (*X*)	CDM (*Y*)	DAM (*X*)	DAM (*Y*)	DM (*X*)	DM (*Y*)	BM (*X*)	BM (*Y*)
1	$\lambda_1 = 1.587$	0.0	−1.256	0.0	−1.256	0.0	−1.265	0.0	−1.425
	$\lambda_2 = 6.168$	0.0	−3.528	0.0	−3.528	0.0	−3.515	0.0	−4.867
	$\lambda_3 = 33.322$	0.0	−11.57	0.0	−11.57	0.0	−10.29	0.0	−10.74
4	$\lambda_1 = 1.565$	0.0	−1.216	0.0	−1.216	0.0	−1.216	0.0	−1.279
	$\lambda_2 = 6.124$	0.0	−3.507	0.0	−3.507	0.0	−3.506	0.0	−3.677
	$\lambda_3 = 23.608$	0.0	−7.229	0.0	−7.229	0.0	−7.224	0.0	−6.152
8	$\lambda_1 = 1.565$	0.0	−1.216	0.0	−1.216	0.0	−1.216	0.0	−1.248
	$\lambda_2 = 6.123$	0.0	−3.506	0.0	−3.506	0.0	−3.506	0.0	−3.579
	$\lambda_3 = 23.590$	0.0	−7.222	0.0	−7.222	0.0	−7.221	0.0	−6.932

Table 3 Eigenvalue sensitivity coefficients of the 19-member frame

Joint	Initial eigenvalue	CDM (*X*)	CDM (*Y*)	DAM (*X*)	DAM (*Y*)	DM (*X*)	DM (*Y*)	BM (*X*)	BM (*Y*)
I	$\lambda_1 = 4.657$	−0.611	0.579	−0.611	0.579	−0.611	0.579	−0.611	0.579
	$\lambda_2 = 33.199$	0.760	2.554	0.760	2.554	0.760	2.553	0.760	2.553
	$\lambda_3 = 73.685$	1.563	3.264	1.563	3.264	1.562	3.263	1.562	3.263
J	$\lambda_1 = 4.657$	0.056	−0.271	0.056	−0.271	0.056	−0.271	0.056	−0.274
	$\lambda_2 = 33.199$	−1.634	5.718	−1.634	5.718	−1.634	5.717	−1.631	5.760
	$\lambda_3 = 73.657$	−10.0	−4.7	−10.0	−4.7	−10.0	−4.69	−9.993	−4.64

Table 4 Computational times of sensitivity analysis with support *I* (CYBER 930 NOS/VE 1.4.1)

	CDM	DAM	DM	BM
CPU, s	19.104	3.197*E*-1	2.736*E*-2	4.228*E*-3
Normalized	4518.45	75.61	6.47	1

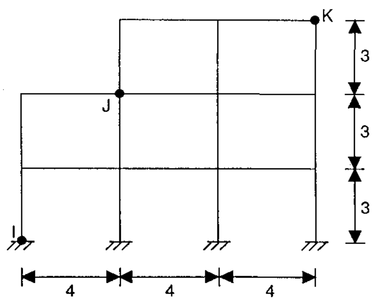

Fig. 9 Nineteen-member frame.

as the design variable. The second columns of these tables list the first three eigenvalues of the example frame. The third and fourth columns give the eigenvalue derivatives approximated by the central difference method (CDM), i.e.,

$$\dot{\lambda} = \frac{\lambda(b + \Delta b) - \lambda(b - \Delta b)}{2\Delta b} \tag{50}$$

where Δb, given a value of 0.005 in., represents the perturbation of the joint location. The central difference method is selected here over the more commonly used forward difference method for the sake of numerical accuracy. The fifth to the tenth columns in these tables list the eigenvalue sensitivity coefficients calculated by Eqs. (42), (32), and (39), respectively. Note that since the structure is symmetric with respect to the movements of joints *J*, *K*, and *L* in the *X* direction, the eigenvalue sensitivities with respect to these movements become zero. This fact is observable in all of the sensitivity analysis methods, as indicated by the columns of zeros in Table 2.

The conclusions drawn for the cantilever beam example can also be made here. For instance, with the coarse mesh, none of the methods yield an acceptable eigenvalue sensitivity of λ_3. In addition, the study of this example confirms that the sensitivity equations, Eqs. (32) and (39), are valid not only for the joint locations but also for the support locations.

Nineteen-Member Frame

A relatively complex structure is proposed in this example to validate the sensitivity equations. The layout of a 19-member frame is given in Fig. 9 along with geometric data. The Young's modulus and the mass density are 10,000.0 psi and 4.0 slug/in.^{3}, respectively. The frame members are solid bars with two different circular sections. The vertical members have a radius of 0.3 in., and the horizontal members have a radius of 0.5 in. The finite element model used here has each member of the structure discretized into four elements. This amounts to 76 elements in total.

The locations of the support I and the joint J are considered as design variables. The numerical results listed in Table 3 generally agree with the conclusions stated in the cantilever beam example. The additional information provided by Table 4 is the computational times (CPU seconds) required by various methods for sensitivity analysis, which are normalized with respect to the computational time of the boundary method. Note that the results presented in Table 4 do not include the CPU time for the eigenvalue analysis of the baseline design.

The results show that the CDM method is the worst of all in terms of computational efficiency. This is because the requirement of two full eigenvalue analyses makes the CDM method extremely expensive to compute, compared with other methods that do not need any eigenvalue analysis to perform the eigenvalue sensitivity analysis. The DAM method, Eq. (42), involves matrix manipulations of all finite elements in the structure whose lengths and orientations are changed because of the movements of joint locations. The DM method, Eq. (32), on the other hand, expresses the eigenvalue sensitivity coefficients in an integral form. Therefore, the computational efforts in the DM method can be significantly reduced by carrying out exact integration. Finally, the sensitivity equation, Eq. (39), of the BM method is represented in a simple algebraic form that involves only terms associated with the moving joints. This is why the BM method is the most efficient one of all.

Conclusions

Two eigenvalue sensitivity equations for a planar frame have been derived in this paper by the continuum approach with variable joint and support locations. Although the derivation process is rather complicated compared with the discrete approach, these equations exhibit the following unique features:

1) These sensitivity equations are capable of giving exact eigenvalue sensitivity coefficients, provided that the exact eigensolutions are available. Even with the approximate eigensolutions of reasonable accuracy, these sensitivity equations are still able to yield accurate sensitivity coefficients.

2) Both sensitivity equations computationally are more efficient than the one derived by the discrete method. In particular, the sensitivity equation derived by the boundary method is extremely fast to compute; however, it is sensitive to the inaccuracy of the eigensolutions.

Appendix

Total Shape Derivatives of Derivatives of Joint Displacements

The total shape derivatives of derivatives of joint displacements (u_i', w_i') can be obtained by taking the material derivatives of Eq. (6) as

$$\begin{bmatrix} \dot{\overline{u_i'}} \\ \dot{\overline{w_i'}} \end{bmatrix} = \dot{\theta}_i \begin{bmatrix} w_i' \\ -u_i' \end{bmatrix} + \begin{bmatrix} \dot{\overline{\upsilon_i'}} \\ \dot{\overline{\omega_i'}} \end{bmatrix} \tag{A1}$$

where $(\dot{\overline{\upsilon_i'}}, \dot{\overline{\omega_i'}})$ are the shape derivatives of (u_i', w_i') with the orientation of the member i being held unchanged, that is,

$$\begin{bmatrix} \dot{\overline{\upsilon_i'}} \\ \dot{\overline{\omega_i'}} \end{bmatrix} = \begin{bmatrix} m_i & n_i \\ -n_i & m_i \end{bmatrix} \begin{bmatrix} \dot{\overline{U_i'}} \\ \dot{\overline{W_i'}} \end{bmatrix} \tag{A2}$$

where $(\dot{\overline{U_i'}}, \dot{\overline{W_i'}})$ can be rewritten in terms of either the total or the relative shape derivatives.[4] Therefore, one has

$$\begin{bmatrix} \dot{\overline{u_i'}} \\ \dot{\overline{w_i'}} \end{bmatrix} = \dot{\theta}_i \begin{bmatrix} w_i' \\ -u_i' \end{bmatrix} + \begin{bmatrix} \dot{\upsilon}_i' \\ \dot{\omega}_i' \end{bmatrix} - v_i' \begin{bmatrix} u_i' \\ w_i' \end{bmatrix} \tag{A3}$$

or

$$\begin{bmatrix} \dot{\overline{u_i'}} \\ \dot{\overline{w_i'}} \end{bmatrix} = \dot{\theta}_i \begin{bmatrix} w_i' \\ -u_i' \end{bmatrix} + \begin{bmatrix} u_{i,\tau}' \\ w_{i,\tau}' \end{bmatrix} + v_i \begin{bmatrix} u_i'' \\ w_i'' \end{bmatrix} \tag{A4}$$

This procedure can also be applied to find the total shape derivatives of the second-order derivative w_i''. Only the final relations are given here, as

$$\begin{aligned} \dot{\overline{w_i''}} &= -u_i''\dot{\theta}_i + \dot{\overline{\omega_i''}} \\ &= -u_i''\dot{\theta}_i + \dot{\omega}_i'' - v_i''w_i' - 2v_i'w_i'' \end{aligned} \tag{A5}$$

or

$$\dot{\overline{w_i''}} = -u_i''\dot{\theta}_i + w_{i,\tau}'' + v_i w_i''' \tag{A6}$$

where the term $\dot{\overline{\omega_i''}}$ is defined by

$$\dot{\overline{\omega_i''}} = -n_i\dot{\overline{U_i''}} + m_i\dot{\overline{W_i''}} \tag{A7}$$

Total Shape Derivative of a Functional Defined by a Line Integral

Based on the definition of material derivatives[4] given in Eq. (16), the total shape derivative of the functional defined in Eq. (20) can be found as

$$\dot{J} = \int_0^l \left(\frac{\partial f}{\partial u}\dot{u} + \frac{\partial f}{\partial w}\dot{w} + \frac{\partial f}{\partial u'}\dot{\overline{u'}} + \frac{\partial f}{\partial w''}\dot{\overline{w''}} + \frac{\partial f}{\partial \tau} + fv' \right) \mathrm{d}s \tag{A8}$$

Next, Eqs. (17), (A3), and (A5) can be used to replace the terms $\dot{u}$, $\dot{w}$, $\dot{\overline{u'}}$, and $\dot{\overline{w''}}$ in terms of $\dot{\theta}$, $\dot{\upsilon}$, $\dot{\omega}$, $\dot{\upsilon}'$, and $\dot{\omega}''$, respectively. The result is stated in Eq. (21). On the other hand, Eqs. (19), (A4), and (A6) can be used to obtain an equation in terms of $\dot{\theta}$, $u_{,\tau}$, $w_{,\tau}$, $u_{,\tau}'$, and $w_{,\tau}''$ as

$$\begin{aligned} \dot{J} = &\int_0^l \Bigg[\frac{\partial f}{\partial \tau} + \left(\frac{\partial f}{\partial u}w - \frac{\partial f}{\partial w}u + \frac{\partial f}{\partial u'}w' - \frac{\partial f}{\partial w'}u'' \right)\dot{\theta} \\ &+ \frac{\partial f}{\partial u}u_{,\tau} + \frac{\partial f}{\partial w}w_{,\tau} + \frac{\partial f}{\partial u'}u_{,\tau}' + \frac{\partial f}{\partial w''}w_{,\tau}'' \Bigg] \mathrm{d}s \\ &+ \int_0^l \Bigg[\left(\frac{\partial f}{\partial u}u' + \frac{\partial f}{\partial w}w' + \frac{\partial f}{\partial u'}u'' + \frac{\partial f}{\partial w''}w''' \right)v \\ &+ fv' \Bigg] \mathrm{d}s \end{aligned} \tag{A9}$$

It is straightforward to show that the integrand in the second integral of Eq. (A9) is exactly identical to $\mathrm{d}(fv)/\mathrm{d}s$. Therefore, integration by parts of the second integral leads to a simpler form. The result is given in Eq. (22).

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